

On local structures of cubicity 2 graphs

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Abstract

A 2-stab unit interval graph (2SUIG) is an axes-parallel unit square intersection graph where the unit squares intersect either of the two fixed lines parallel to the X -axis, distance $1 + \epsilon$ ($0 < \epsilon < 1$) apart. This family of graphs allow us to study local structures of unit square intersection graphs, that is, graphs with cubicity 2. The complexity of determining whether a tree has cubicity 2 is unknown while the graph recognition problem for unit square intersection graph is known to be NP-hard. We present a polynomial time algorithm for recognizing trees that admit a 2SUIG representation.

Keywords: cubicity, geometric intersection graph, unit square intersection graph, 2-stab unit interval graph.

1 Introduction

Cubicity $cub(G)$ of a graph G is the minimum d such that G is representable as a geometric intersection graph of d -dimensional (axes-parallel) cubes [6]. The notion of cubicity is a special case of boxicity [6]. Boxicity $box(G)$ of a graph G is the minimum d such that G is representable as a geometric intersection graph of d -dimensional (axes-parallel) hyper-rectangles. Given a graph G it is NP-hard to decide if $box(G) \leq n$ [5] and $cub(G) \leq n$ [3] for all $n \geq 2$. On the other hand, the family of graphs with boxicity 1 and the family of graphs with cubicity 1 are just the families of interval and unit interval graphs, respectively. The graph recognition problems for both these families are solvable in polynomial time.

Trees with boxicity 1 are the caterpillar graphs while all trees have boxicity 2. On the contrary, determining cubicity of a tree seems to be a more difficult problem. It is easy to note that trees with cubicity 1 are paths. For higher dimensions, Babu et al. [1] presented a randomized algorithm that runs in polynomial time and computes cube representations of trees, of dimension within a constant factor of the optimum. The complexity of determining the cubicity of a tree is unknown [1].

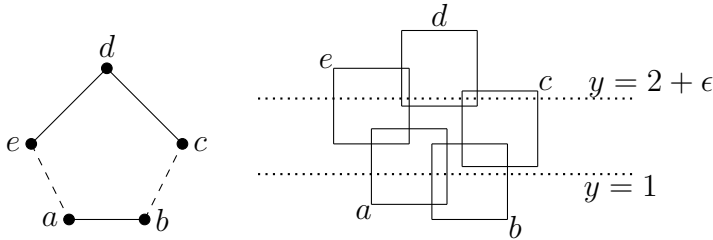


Figure 1: A representation (right) of a *2SUIG* graph (left).

In a recent work [2], some new families of graphs, based on the local structure of boxicity 2 and cubicity 2 graphs were introduced and studied. A *2-stab unit interval graph (2SUIG)* is an axes-parallel unit square intersection graph where the unit squares intersect either of the two fixed lines, called upper and lower stab lines, parallel to the X -axis, distance $1 + \epsilon$ ($0 < \epsilon < 1$) apart (see Fig. 1 for example). For convenience, let $y = 1$ be the *lower stab line* and $y = 2 + \epsilon$ be the *upper stab line* where $\epsilon \in (0, 1)$ is a constant, for the rest of the article. The family of such graphs are called the *2SUIG* family, introduced [2] for studying the local structures of cubicity 2 graphs. A geometric representation of the above mentioned type of a graph is called a *2SUIG representation* of the graph. Given a 2SUIG representation of a graph G , the vertices corresponding to the unit squares intersecting the upper stab line are called *upper* vertices and the vertices corresponding to the unit squares intersecting the lower stab line are called *lower* vertices. If a set of vertices are all lower (or upper) vertices then we say they are in the *same stab*.

In this article, we characterize all trees that admit a 2SUIG representation using forbidden structures. In particular, we prove the following:

Theorem 1.1. *Determining whether a given tree $T = (V, E)$ is a 2SUIG can be done in $O(|V|)$ time.*

If a tree is a 2SUIG, then our algorithm can be used to find a 2SUIG representation of it. Our algorithm finds a forbidden structure responsible for the tree not having a 2SUIG representation. In particular, 2SUIG is a graph family with cubicity (and boxicity) 2 which contains the family of unit interval graphs as a subfamily. Moreover, the family of 2SUIG graphs is not perfect [4] as 5-cycle has a 2SUIG representation (see Fig. 1). So our work, to the best of our knowledge, is the first non-trivial work on recognizing subclass of trees with cubicity 2.

2 Preliminaries

To prove our results we will need several standard and non-standard definitions which we will present in this section.

Let G be a graph. The set of vertices and edges are denoted by $V(G)$ and $E(G)$, respectively. A vertex subset I of G is an *independent set* if all the vertices of I are pairwise non-adjacent. The cardinality of the largest independent set of G is its *independence number*, denoted by $\alpha(G)$.

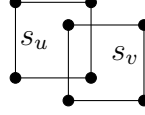


Figure 2: In the above picture $s_u <_x s_v$ and $s_u <_y s_v$.

Let G be a unit square intersection graph with a fixed representation R . We denote the unit square in R corresponding to the vertex v of G by s_v . In this article, by a unit square we will always mean a closed unit square. The co-ordinates of the left lower-corner of s_u is denoted by (x_u, y_u) . Given a graph G with a 2SUIG representation R and two vertices $u, v \in V(G)$ we say $s_u <_x s_v$ if $x_u < x_v$ and $s_u <_y s_v$ if $y_u < y_v$ (see figure 2). Let H be a connected subgraph of G . Consider the union of intervals obtained from the projection of unit squares corresponding to the vertices of H on x -axis. This so obtained interval $span(H)$ is called the *span* of H in R .

A *leaf* is a vertex with degree 1. A *caterpillar* is a tree where every leaf vertex is adjacent to a vertex of a fixed path. A *branch vertex* is a vertex having degree more than 2. A *branch edge* is an edge incident to a branch vertex. A *claw* is the complete bipartite graph $K_{1,3}$. Given a 2SUIG representation R of a graph G , an edge uv is a *bridge edge* if s_u and s_v intersect different stab lines.

Let $P = v_1 v_2 \dots v_k$ be a path with a 2SUIG representation R . The path P is a *monotone path* if either $s_{v_1} <_x s_{v_2} <_x \dots <_x s_{v_k}$ or $s_{v_k} <_x s_{v_{k-1}} <_x \dots <_x s_{v_1}$. Let $P = v_1 v_2 \dots v_k$ be a monotone path with $s_{v_1} <_x s_{v_2} <_x \dots <_x s_{v_k}$ and all v_i 's are in the same stab. Observe that $\alpha(P) = \lceil \frac{k}{2} \rceil < span(P) \leq k$ are tight bounds for $span(P)$. Fix some constant $c \in (0, .5)$ for the rest of this article. A monotone representation of P is *stretched* if $span(P) = k$ and is *shrunked* if $span(P) = \lceil \frac{k}{2} \rceil + c$ (see Fig. 3). The value of c will not affect our proof. If all the vertices of P are in the same stab, then P must be monotone. So, a path like P can have four different such 2SUIG representations:

1. *lower-right monotone*: all the vertices of the path are lower vertices with $s_{v_1} <_x s_{v_2} <_x \dots <_x s_{v_k}$;
2. *upper-right monotone*: all the vertices of the path are upper vertices with $s_{v_1} <_x s_{v_2} <_x \dots <_x s_{v_k}$;
3. *lower-left monotone*: all the vertices of the path are lower vertices with $s_{v_k} <_x s_{v_{k-1}} <_x \dots <_x s_{v_1}$;
4. *upper-left monotone*: all the vertices of the path are upper vertices with $s_{v_k} <_x s_{v_{k-1}} <_x \dots <_x s_{v_1}$.

The first two types of monotone paths are called *right monotone* and last two types are called *left monotone*.

A path $P = v_1 v_2 \dots v_k$ is called a *folded path* if it has a degree two vertex u such that either $s_u <_x s_v$ for all $v \in V(P) \setminus \{u\}$ or $s_v <_x s_u$ for all $v \in V(P) \setminus \{u\}$.

A *red edge* of a tree T is an edge e such that each component of $T \setminus \{e\}$ contains a claw. A *red path* is a path induced by red edges. A *maximal red path* is a red path that

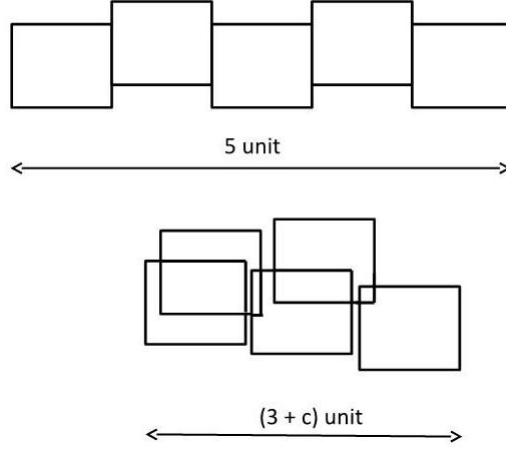


Figure 3: A stretched (above) and shrunk (below) representation of a path on five vertices.

is not properly contained in another red path. Let $P = v_1v_2\dots v_k$ be a maximal red path in T . The vertices v_1 and v_k are *endpoints* of P .

3 Proof of Theorem 1.1

Given a tree T , here our objective is to determine whether T has a 2SUIG representation. Let $T = (V, E)$ be a fixed tree. We will now prove several necessary conditions for T being a 2SUIG. On the other hand, we will show that these conditions together are also sufficient and can be verified in $O(|V|)$ time.

3.1 Structural properties

We will start by proving some structural properties of T assuming it is a 2SUIG.

Lemma 3.1. *If T has a 2SUIG representation, then its vertices have degree at most four.*

Proof. Let u be a vertex of T . Any unit square intersecting s_u must contain one of the four corners of s_u . Therefore, if u has degree greater than four, then at least two unit squares will contain the same corner of s_u creating a 3-cycle. \square

Note that the above lemma can easily be extended for trees that are unit square intersection graphs.

A tree T that admits a 2SUIG representation does not necessarily have a red edge. But if T has at least one red edge then the red edges of T must induce a path.

Lemma 3.2. *If T has a 2SUIG representation, then either T has no red edge or the set of red edges of T induces a connected path.*

Proof. Let T has at least one red edge and T' be the graph induced by all red edges. First we will show that T' is connected. Thus, assume that T' has at least two components T_1 and T_2 . Then there is a path P in T connecting T_1 and T_2 . Note that removing an edge e of P creates two components of T each of which contains a claw. Thus, e should be a red edge. Therefore, T' is connected.

Now we will show that T' is a path. Assume that v is a vertex of T' with degree at least 3. Also, let v_1, v_2 and v_3 be three neighbors of v in T' . In any 2SUIG representation of T , at least three corners of s_v must be intersected by s_{v_1}, s_{v_2} and s_{v_3} . Without loss of generality, we assume that s_{v_1} intersects the upper-left corner of s_v , s_{v_2} intersect the upper-right corner of s_v , and s_{v_3} intersects the left lower-corner of s_v . This implies that s_{v_1}, s_{v_2} intersect the upper stab line while s_{v_3} intersects the lower stab line.

Note that a claw has a 2SUIG representation. Any such representation of a claw will have squares intersecting the upper stab line and squares intersecting the lower stab line. As each component of $T \setminus \{vv_1\}$ has a claw, there must be a path of the form $v_1v_{11}v_{12}\dots v_{1k}$ in T such that $s_{v_{1i}} <_x s_{v_1}$ for all $i \in \{1, 2, \dots, k\}$ where $s_{v_{1k}}$ intersects the lower stab line. Similarly, as each component of $T \setminus \{vv_2\}$ has a claw, there must be a path of the form $v_2v_{21}v_{22}\dots v_{2k'}$ in T such that $s_{v_2} <_x s_{v_{2i}}$ for all $i \in \{1, 2, \dots, k'\}$ where $s_{v_{2k'}}$ intersects the lower stab line. Moreover, as each component of $T \setminus \{vv_3\}$ has a claw, there must be a path of the form $v_3v_{31}v_{32}\dots v_{3k''}$ in T where $s_{v_{3k''}}$ intersects the upper stab line. This will force a cycle in the representation of T , a contradiction. Thus, T' must be a path. \square

The above result leads us to two cases: when T has a red path and when T does not have any red path. Note that by Lemma 3.2, if T is a 2SUIG, then either T has no red edge or the red edges induces a path in T . If the red edges of T induces a path P , then construct the *extended red path* $A = a_1a_2\dots a_k$ by including the edge(s), that are not red, incident to the endpoint(s) of P that have degree two in T . In particular, if both the end points of P are branch vertices, the extended red path $A = P$. On the other hand, if T has no red edges, then distance between any two branch vertices is at most 2. Thus, there exist a vertex v in T whose closed neighborhood $N[v]$ contains all the branch vertices of T . Choose (if not found to be unique) one such special vertex v . If v has degree two then consider the path uvw induced by the closed neighborhood of v and call it the extended red path of T . If v does not have degree two, then the extended red path of T is the singleton vertex v . In any case, rename the vertices of the extended red path $A = a_1a_2\dots a_k$ so that we can speak about it in an uniform framework along with the case T having red edges. We fix such an extended red path $A = a_1a_2\dots a_k$ for the rest of this article. The vertices $V_A = \{a_1, a_2, \dots, a_k\}$ of this extended red path A are called the *red vertices*.

Lemma 3.3. *If T has a 2SUIG representation and does not have any red edge, then the number of branch vertex in T is at most 5.*

The above follows directly from the fact that the vertices of T has degree at most four.

Lemma 3.4. *If T has at most one branch vertex with degree at most four, then T has a 2SUIG representation.*

Proof. If T has no branch vertices, then T is a path which admits a unit interval representation. On the other hand, T with one branch vertices of degree at most four is a subdivision of $K_{1,3}$ or $K_{1,4}$. These graphs clearly admit 2SUIG representation. \square

The next result will provide intuition about how a tree having a 2SUIG representation looks like.

Lemma 3.5. *A branch vertex of a tree T is either a red vertex or is adjacent to a red vertex.*

Proof. If T has no red edges, then there exists a red vertex v in T such that all the branch vertices are in the closed neighborhood $N[v]$ of v .

Thus suppose that the red edges of T induces a path. Let a red vertex v and a non-red branch vertex u be connected by a path with no red edges of length at least 2. Clearly after deleting this path, the component containing v contains a claw. Thus if we delete the edge $\{e\}$ of the path incident to v , then both the components of $T \setminus \{e\}$ contains a claw, a contradiction. \square

Note that if T is a 2SUIG tree with at least two branch vertices, then the endpoints a_1 and a_k of the extended path A must be branch vertices of T . Assume that $A_b = \{a_1 = a_{i_1}, a_{i_2}, \dots, a_{i_{k'}} = a_k\}$ be the branch vertices of A where $1 = i_1 < i_2 < \dots < i_{k'} = k$.

The neighbors of red vertices that are not red are called *agents*. An agent v is adjacent to exactly one red vertex, say a_j , of T . We call v is an agent of a_j in this case.

If we delete all the red vertices and agents from a 2SUIG tree T then by Lemma 3.5 we will be left with some disjoint paths. Each such path actually starts from (that is, is one endpoint of) one of the agents. Let $P = v_1 v_2 \dots v_l$ be a path where v_1 is an agent and the other vertices are neither agent nor red vertices. Also v_2, v_3, \dots, v_{l-1} are degree 2 vertices and v_l is a leaf. Then the path $P' = v_2 v_3 \dots v_{l-1} v_l$ is called a *tail* of agent v_1 . Let v_1 be an agent of the red vertex a_j . Sometimes we will also use the term “tail P' of the red vertex a_j ”. Deleting the tail P' is to delete all the vertices of P' . The red vertex a_j , all its agents and tails are together called *a_j and its associates* for each $j \in \{1, 2, \dots, k\}$. Note that an agent has exactly two tails by allowing tails with zero vertices. Let us set the following conventions: the tails of an agent z are the *long tail* $lt(z)$ and the *short tail* $st(z)$ such that $|lt(z)| \geq |st(z)|$ where $|lt(z)|$ and $|st(z)|$ denotes the number of vertices in the respective tails. Now we have enough nomenclatures (see Fig. 4) to present the rest of the proof.

Observe that if we delete all the tails of a 2SUIG tree T , then we are left with a caterpillar with maximum degree at most 4. This is an interesting observation as we know that a tree is an interval graph if and only if it is a caterpillar graph.

3.2 Partial description of the canonical representation

Till now we have proved some structural properties of a 2SUIG tree. Now we will discuss about the structure of its 2SUIG representation. In the following, we will show that there is a canonical way to represent a 2SUIG tree. First we will describe the representation of the extended red path followed by representation of the agents and their tails.

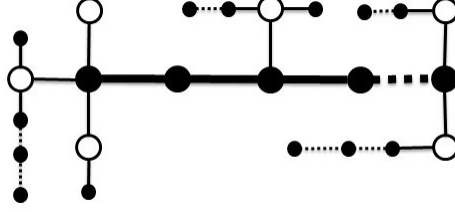


Figure 4: The nomenclature – red edges = thick lines; new edge(s) added to the red path for obtaining the extended red path = thick dotted line(s); red vertices = big solid circles, agents = big hollow circles, tails = thin dotted lines; vertices of the tails = small solid circles.

Lemma 3.6. *If T is a 2SUIG with at least one red edge, then there exists a 2SUIG representation where the extended red path of T is monotone.*

Proof. Let A be the extended red path of T with a representation R . If A is not monotone then one of the following is true. (i) A is a folded path. (ii) There are three vertices $\{a_i, a_{i+1}, a_{i+2}\} \in V(A)$ such that $s_{a_{i+1}} <_x s_{a_i}$ and $s_{a_{i+1}} <_x s_{a_{i+2}}$ where $i > 1$ and for all $j < i$ we have $s_{a_j} <_x s_{a_{j+1}}$.

(i) There is a vertex $u \in V(A)$ with $s_v <_x s_u$ for all $v \in V(A) \setminus \{u\}$. Then there will be two claws C_1, C_2 in two different components of $T \setminus \{u\}$ with $s_v <_x s_u$ for all $v \in V(C_1) \cup V(C_2)$. But as T is a 2SUIG, this configuration will force a cycle in it. This is a contradiction.

(ii) Without loss of generality assume s_{a_i} intersects the lower stab line. Then $s_{a_{i+1}}$ and $s_{a_{i+2}}$ must intersect the upper stab line. Let T_{i+1} be the component of T obtained by deleting the edge $a_i a_{i+1}$ and contains a_{i+1} . There is a claw C_3 in T_{i+1} with $s_{a_{i+1}} <_x s_w$ for all $w \in V(C_3)$. Thus, an agent z of a_i with $s_{a_i} <_x s_z$ is not a branch vertex as otherwise this will force a cycle. Hence its tail can be presented by a lower-right monotone representation. Therefore, we can translate (rigid motion) the component T_{i+1} to the right to obtain a 2SUIG representation of T where the extended red path A is monotone. \square

Similarly it can be shown that if T is a 2SUIG with no red edge, still it admits a representation where the extended red path A is monotone. The proof can be argued in a way similar to the proof of Lemma 3.6.

Now we will show that it is possible to stretch the monotone extended red path without much problem. The following result is also applicable for trees without red edges.

Lemma 3.7. *If T admits a 2SUIG representation R with a monotone extended red path, then there exists a 2SUIG representation where A is stretched.*

Proof. Let $e = a_i a_{i+1}$ be an edge of the extended red path A with $x_{a_i} < x_{a_{i+1}}$ in R . Let T_i and T_{i+1} be the components of $T \setminus \{e\}$ containing a_i and a_{i+1} , respectively. Now translate (rigid motion) the component T_{i+1} to the right obtaining a 2SUIG representation with $x_{a_{i+1}} = x_{a_i} + 1$. We are done by performing this operation on every edge of A . \square

We turn our focus on the bridge edges of the extended red path.

Lemma 3.8. *If T admits a 2SUIG representation R with a stretched monotone extended red path, then there exists a 2SUIG representation where every red bridge vertex is a branch vertex.*

Proof. Let $A = a_1 a_2 \dots a_k$ be a right monotone extended red path of T with respect to R . Let $e = a_i a_{i+1}$ be a bridge edge of the extended red path A where a_{i+1} is not a branch vertex. Also assume that a bridge vertex a_j is a branch vertex for all $j < i$.

Let $S = \{v \in V(T) | s_{a_i} <_x s_v\}$. Let T' be the graph induced by S . Note that S contains the vertex a_{i+1} . Now consider the reflexion of the 2SUIG representation of T' induced by R with respect to the X -axis. This will give us a picture where every unit square corresponding to the vertices of T' lies under the X -axis. This is a particular unit square representation R' of T' which in fact is also a 2SUIG representation of it if we consider the stab lines to be $y = -1$ and $y = -2 - \epsilon$. In this 2SUIG representation R' , the lower vertices with respect to R of T' became upper vertices and vice versa. Now translate the unit square representation R' upwards until all the upper vertices (with respect to R') of T' intersects $y = 2 + \epsilon$, all the lower vertices intersect $y = 1$. Note that a_i can have at most one degree 2 agent in T' and thus, that agent can have at most one tail. After what we did above, we can adjust the Y -co-ordinates of that agent and its tail, if needed, to obtain a 2SUIG representation of T .

We will be done by induction after handling one more case. The case where a_i is the first bridge vertex of A which is not a branch vertex while a_{i+1} is a branch vertex. Let $S' = \{v \in V(T) | s_v <_x s_{a_{i+1}}\}$ and let T'' be the graph induced by S' . To achieve our goal, we do the exact same thing with T'' that we did with T . This will provide us a 2SUIG representation of T where each bridge vertex a_j is a branch vertex for all $j \leq i + 1$. Hence we are done by induction. \square

After proving that the extended red path is stretched and monotone we will prove the opposite for tails in the following sense:

Lemma 3.9. *If T admits a 2SUIG representation R , then there exists a 2SUIG representation where each tail is a shrunk monotone path and all its vertices are in the same stab.*

Proof. Let $P = v_2 v_3 \dots v_{l-1}$ be a tail of agent v_1 . Note that if all vertices of the tail P are in the same stab then P must be monotone. Furthermore, if P is not shrunk in R then we can shrink it to obtain a new representation of T without changing anything else of R .

Therefore, to complete the proof, let us assume that not all vertices of P are in the same stab. Then at least one edge e of P is a bridge edge. Any bridge edge divides the stab lines into two parts, left and right. Assume, without loss of generality, that s_{v_1} is in

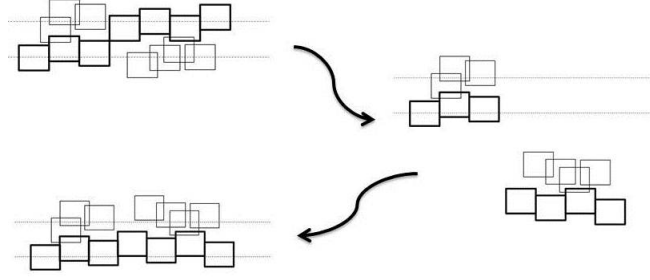


Figure 5: Explaining the reflection and translation described in the proof of Lemma 3.8 with an example.

the left part. Thus, as there are no branch vertices in the tail, we do not have any vertex $w \in V(T) \setminus V(P)$ with s_w lying in the right part. Thus, we can modify the representation R by placing all the vertices of the tail P in the same stab making use of the empty right part. \square

3.3 Properties of the canonical representation

According to the preceding discussions, we know that if T is a 2SUIG, then it admits a representation where its extended red path is a monotone stretched path. Without loss of generality, assume that R is such a 2SUIG representation of T . The other vertices of T are the agents and the vertices of the tails. Note that the endpoints a_1 and a_k can have at most 3 agents and 6 tails while the other red vertices can have at most 2 agents and 4 tails.

Lemma 3.10. *Let T be a 2SUIG tree with a representation R where the extended red path A is a stretched right monotone path and R satisfies the conditions of Lemma 3.8 and 3.9.*

- (a) *If each right monotone tail of T is such that it is not possible to make the tail left monotone and obtain a 2SUIG representation of T from R without changing anything else, then any red vertex other than a_k has at most one right monotone tail having at least two vertices.*
- (b) *If each left monotone tail of T is such that it is not possible to make the tail right monotone and obtain a 2SUIG representation of T from R without changing anything else, then any red vertex other than a_1 has at most one left monotone tail having at least two vertices.*

Proof. (a) Let a_j ($j < k$) be a red vertex of T with at least two monotone tails. Without loss of generality, assume that a_j is a lower vertex. Note that $s_{a_{j+1}}$ contains either the upper-right corner or the lower-right corner of s_{a_j} . Let v be an agent of a_j .

- Case 1. If s_v contains the lower-left corner of s_{a_j} , then v cannot have a right monotone tail for any $j \geq 2$ as $s_{a_{j-1}}$ contains the upper-left corner of s_{a_j} . If $j = 1$, then a right monotone tail $P = v_1 v_2 \dots v_l$ of v must be upper-right monotone. This means vv_1 is a bridge edge. This will mean, there is no upper vertex z in T with $s_z <_x s_{v_1}$. Thus, if we make the tail P an upper-left monotone tail instead (we keep the position of s_{v_1} as before but change the positions of the other vertices of P), then we obtain a 2SUIG representation of T . But this should not be possible according to the assumptions. Hence v cannot have a right-monotone tail for any red vertex a_j ($j \neq k$).
- Case 2. If s_v contains the upper-left corner of s_{a_j} , then v can indeed have an upper-right monotone tail. Note that, in this case, v can have at most one right monotone path, an upper-right monotone that is. If some other neighbor of a_j contains the upper-right corner of s_{a_j} , then the right monotone tail of v can have at most one vertex in order to avoid cycles in T .
- Case 3. If s_v contains the upper-right corner of s_{a_j} , then v can have at most one right monotone tail, an upper-right monotone tail to be specific, as this situation implies that $s_{a_{j+1}}$ contains the lower-right corner of s_{a_j} .

Therefore, only the agents containing upper-left corner or upper-right corner of s_{a_j} can have at most one right monotone tail each. But if both types of agents are present, then the right monotone tail of the agent containing upper-left corner of s_{a_j} can have at most one vertex.

(b) This proof can be done similarly like (a). □

From the above we can infer the following:

Lemma 3.11. *Let T be a 2SUIG tree with a representation R where a lower (or upper) red vertex has two upper (or lower) neighbors.*

1. *If the left neighbor is an agent then it can have at most one right monotone tail having at most one vertex.*
2. *If the right neighbor is an agent then it can have at most one left monotone tail having at most one vertex.*
3. *If both the neighbors are agents then the left neighbor can have a right monotone tail and the right neighbor can have a left monotone tail having one vertex each.*

In the above lemma, the third case is to say that the worst case scenarios of the first two cases can take place simultaneously. Now we will discuss the length of the tails that can be accommodated between two red branch vertices.

Lemma 3.12. *Let T be a 2SUIG tree with a representation R with a stretched right monotone extended red path A . Let a_l, a_{l+m} ($m = 0$ is possible) be two red branch vertices such that a_l has a right monotone tail P and a_{l+m} has a left monotone tail P' in the same stab with no vertex v of T satisfying $s_p <_x s_v <_x s_{p'}$ where p, p' are the leaf vertices of the tails P, P' , respectively. Then, depending on the positions of the corresponding agents v of P and v' of P' , R must satisfy one of the following conditions:*

1. If $s_{a_l} <_x s_v$ and $s_{v'} <_x s_{a_{l+m}}$, then $\alpha(P_v) + \alpha(P_{v'}) \leq m$;
2. If $s_v <_x s_{a_l}$ and $s_{v'} <_x s_{a_{l+m}}$, then $\alpha(P_v) + \alpha(P_{v'}) - 1 \leq m$;
3. If $s_v <_x s_{a_l}$ and $s_{a_{l+m}} <_x s_{v'}$, then $\alpha(P_v) + \alpha(P_{v'}) - 2 \leq m$;
4. If $s_{a_l} <_x s_v$ and $s_{a_{l+m}} <_x s_{v'}$, then $\alpha(P_v) + \alpha(P_{v'}) - 1 \leq m$;

where P_v is the path induced by $V(P) \cup \{v\}$ and $P_{v'}$ is the path induced by $V(P') \cup \{v'\}$.

Proof. Let $A' = a_l a_{l+1} \dots a_{l+m}$ and $s_v <_x s_{a_l}$ and $s_{a_{l+m}} <_x s_{v'}$. As A' is stretched and P_v and $P_{v'}$ are both shrunk we have $\text{span}(A') + 2 \geq \text{span}(P_v) + \text{span}(P_{v'})$. This implies $m + 1 + 2 = m + 3 \geq \lceil \alpha(P_v) + \alpha(P_{v'}) + 2c \rceil = \alpha(P_v) + \alpha(P_{v'}) + 1$ and hence condition 3. \square

The other conditions can be proved similarly. \square

In our prescribed representation of T , assuming it is a 2SUIG, bridge edges of the extended red path are induced by red branch vertices. Here we show that if two adjacent red vertices have degree 4 each, then they must be branch vertices.

Lemma 3.13. *If two adjacent red vertices both have degree 4, then they must be in different stabs.*

Proof. Without loss of generality assume that $a_l a_{l+1}$ is such an edge where a_l, a_{l+1} are both degree 4 lower vertices with $s_{a_l} <_x s_{a_{l+1}}$. Then either the upper-right corner of s_{a_l} is contained in $s_{a_{l+1}}$ or the upper-left corner of $s_{a_{l+1}}$ is contained in s_{a_l} . If the upper-right corner of s_{a_l} is contained in $s_{a_{l+1}}$, then a_l cannot have more than three neighbors as no neighbor other than $s_{a_{l+1}}$ can contain one of the right corners of s_{a_l} . We can argue similarly for the other case as well. \square

3.4 The canonical representation

In this section suppose that T is a tree with maximum degree 4 such that either there is no red edge or the red edges induces a path. We will try to obtain a 2SUIG representation of T and if our process fails to obtain such a presentation, then we will conclude that T is not a 2SUIG. Also assume that the extended red path of T is $A = a_1 a_2 \dots a_k$. Due to Lemma 3.6 and 3.7 we can assume that A is a stretched right monotone path and a_1 is a lower vertex.

Our strategy is to first represent a_1 and its associates and then to represent a_i and its associates one by one in accending order of indices where $i \in \{2, 3, \dots, k\}$. In each step our strategy is to represent a_i and its associates in such a way that the maximum value of x_v is minimized where v is a vertex from a_i and its associates.

Note that the main difficulty is to represent a_i and its associates when $d(a_i) \geq 3$ as otherwise a_i do not have any agents or tails. We start off with the following useful lemma.

Lemma 3.14. *There exists a 2SUIG representation, satisfying all the properties of the canonical representation proved till now, with a_1 and a_2 in the same stab if and only if either $d(a_1) \leq 3$ or $d(a_2) \leq 3$.*

Proof. The “only if” part follows from Lemma 3.13. The “if” part can be proved similarly like the proof of Lemma 3.8. \square

The above result completely determines when a_2 will be in the lower stab and when it will be in the upper stab.

3.4.1 Representation of a_1 and its associates when $k \neq 1$

First we will handle the case $k \neq 1$. Now we are going to list out the way to obtain the canonical representation of a_1 and its associates and the conditions for it to be valid through case analysis. Also in any representation the agents intersecting the lower-left corner, the upper-left corner, the upper-right corner and the lower-right corner of s_{a_1} are renamed as z_1, z_2, z_3 and z_4 , respectively. The conditions below are simple conditions for avoiding cycles in the graph.

Case 1: $d(a_1) = 4, d(a_2) = 4$. In this case a_2 is an upper vertex by Lemma 3.13, s_{a_2} intersects the upper-right corner of s_{a_1} and the three agents of a_1 are z_1, z_2, z_4 .

- (1) $lt(z_1)$ is shrinked lower-left monotone and $st(z_1)$ is shrinked upper-left monotone.
- (2) $lt(z_2)$ is shrinked upper-left monotone and $st(z_2)$ is shrinked upper-right monotone.
- (3) $|st(z_2)| \leq 1$ and if $|st(z_1)| > 0$, then $|lt(z_2)| \leq 1$.
- (4) $|lt(z_4)| = 0$.

Case 2: $d(a_1) = 3, d(a_2) = 4$. In this case a_2 is a lower vertex by Lemma 3.14, s_{a_2} intersects the upper-right corner of s_{a_1} and the two agents of a_1 are z_1, z_2 .

- (1) conditions (1)–(3) of Case 1.

Case 3: $d(a_1) = 4, d(a_2) = 3$. In this case a_2 is a lower vertex by Lemma 3.14, s_{a_2} intersects the lower-right corner of s_{a_1} and the three agents of a_1 are z_1, z_2, z_3 .

- (1) condition (1)–(3) from Case 1.
- (2) $|lt(z_3)| \leq 1$, $lt(z_3)$ is shrinked upper-left monotone and $st(z_3)$ is shrinked upper-right monotone.

Case 4: $d(a_1) = 3, d(a_2) = 3$. In this case a_2 is a lower vertex by Lemma 3.14, s_{a_2} intersects the lower-right corner of s_{a_1} and the two agents of a_1 are z_1, z_2 .

- (1) $lt(z_1)$ is shrinked lower-left monotone and $st(z_1)$ is shrinked upper-left monotone.
- (2) if $|st(z_1)| > 0$, then $lt(z_2)$ is shrinked upper-right monotone with $|lt(z_2)| \leq 3$ and $st(z_2)$ is shrinked upper-left monotone $|st(z_2)| \leq 1$.
- (3) if $|st(z_1)| = 0$, then $lt(z_2)$ is shrinked upper-left monotone and $st(z_2)$ is shrinked upper-right monotone $|st(z_2)| \leq 3$.

Case 5: $d(a_1) = 4, d(a_2) = 2$. In this case a_2 is a lower vertex by Lemma 3.14, s_{a_2} intersects the lower-right corner of s_{a_1} and the three agents of a_1 are z_1, z_2, z_3 .

- (1) condition (1)–(3) from Case 1.
- (2) if $|lt(z_3)| \leq 1$, then $lt(z_3)$ is shrinked upper-left monotone and $st(z_3)$ is shrinked upper-right monotone.
- (3) if $|lt(z_3)| > 1$, then $|st(z_3)| \leq 1$ and $lt(z_3)$ is shrinked upper-right monotone and $st(z_3)$ is shrinked upper-left monotone.

Case 6: $d(a_1) = 3, d(a_2) = 2$. In this case a_2 is a lower vertex by Lemma 3.14, s_{a_2} intersects the lower-right corner of s_{a_1} and the two agents of a_1 are from $\{z_1, z_2, z_3\}$.

- (1) $lt(z_1)$ is shrinked lower-left monotone and $st(z_1)$ is shrinked upper-left monotone.
- (2) if both z_1, z_2 exists and $|st(z_1)| = 0$, then $lt(z_2)$ is shrinked upper-left monotone and $st(z_2)$ is shrinked upper-right monotone.
- (3) if both z_1, z_2 exists, $|st(z_1)| > 0$ and $|lt(z_2)| \leq 1$, then $lt(z_2)$ is shrinked upper-left monotone and $st(z_2)$ is shrinked upper-right monotone.
- (4) if both z_1, z_2 exists, $|st(z_1)| > 0$ and $|lt(z_2)| > 1$, then $st(z_2)$ is shrinked upper-left monotone and $lt(z_2)$ is shrinked upper-right monotone with $|st(z_1)| \leq 1$.
- (5) there is no case when both z_2, z_3 exists as we can always modify this representation by making the agent playing the role of z_2 play the role of z_1 instead.
- (6) there is no case when both z_1, z_3 exists with $|st(z_1)| = 0$ as we can always modify this representation by making the agent playing the role of z_3 play the role of z_2 instead.
- (7) there is no case when both z_1, z_3 exists with $|st(z_3)| \leq 1$ and $|lt(z_3)| \leq 3$ as we can always modify this representation by making the agent playing the role of z_3 play the role of z_2 instead.
- (8) if both z_1, z_3 exists, $|st(z_1)| > 0$ and $|lt(z_3)| \leq 3$, then $lt(z_3)$ is shrinked upper-left monotone and $st(z_2)$ is shrinked upper-right monotone.
- (9) if both z_1, z_3 exists, $|st(z_1)| > 0$ and $|lt(z_3)| > 3$, then $st(z_3)$ is shrinked upper-left monotone with $|st(z_1)| \leq 3$ and $lt(z_2)$ is shrinked upper-right monotone.

The square s_{a_2} must intersect one of the right corners of s_{a_1} . In each of the cases listed above, there can be at most $3! = 6$ possible ways of in which the agents of a_1 can play the role of z_1, z_2, z_3, z_4 . Among all possible ways those which satisfies the above conditions, we choose the one for which the leaf of the right-monotone tail of z_2 (only when z_3, z_4 does not exist) or z_3 (z_4 cannot exist) or z_4 (z_3 cannot exist) is minimized with respect to $<_x$. As there are at most a constant number of probes to be made, this is achievable in constant time. Moreover, such a representation, if found, will be called the *optimized representation of a_1 and its associates*. Otherwise, T is not a 2SUIG.

3.4.2 Representation of a_1 and its associates when $k = 1$

Now we will handle the case $k = 1$. Let a_1 be a lower vertex and the agents intersecting the lower-left corner, the upper-left corner, the upper-right corner and the lower-right corner of s_{a_1} are renamed as z_1, z_2, z_3 and z_4 , respectively. The agents should follow the conditions listed below. These are simple conditions for avoiding cycles.

- (1) P_{11} is shrunk lower-left monotone and P_{12} is shrunk upper-left monotone where $\{P_{11}, P_{12}\} = \{st(z_1), lt(z_1)\}$.
- (2) P_{21} is shrunk upper-left monotone and P_{22} is shrunk upper-right monotone where $\{P_{21}, P_{22}\} = \{st(z_2), lt(z_2)\}$.
- (3) P_{31} is shrunk upper-left monotone and P_{32} is shrunk upper-right monotone where $\{P_{31}, P_{32}\} = \{st(z_3), lt(z_3)\}$.
- (4) P_{41} is shrunk lower-right monotone and P_{42} is shrunk upper-right monotone where $\{P_{41}, P_{42}\} = \{st(z_4), lt(z_4)\}$.
- (5) either $|P_{12}| = 0$ or $|P_{21}| \leq 1$.
- (6) if z_2 does not exist, then either $|P_{12}| = 0$ or $|P_{31}| \leq 3$.
- (7) if both z_2, z_3 exist, then $|P_{22}|, |P_{31}| \leq 1$.
- (8) either $|P_{42}| = 0$ or $|P_{32}| \leq 1$.
- (9) if z_3 does not exist, then either $|P_{42}| = 0$ or $|P_{22}| \leq 3$.

As there are at most a constant number of probes to be made, this is achievable in constant time. For this special case too, such a representation, if found, will be called the *optimized representation of a_1 and its associates*. Otherwise, T is not a 2SUIG.

3.4.3 Representation of a_i and its associates for all $1 < i < k$

Now we will inductively describe the canonical representation of a_i and its associates given the canonical representation of a_j and its associates for all $j < i$ and the conditions for it to be valid through case analysis. Note that as the way of having the canonical representation of a_1 and its associates is known, the conditions listed below is readily applicable for finding the canonical representation of a_2 and its associates. Furthermore, it is applicable for finding the canonical representation of a_i and its associates by induction for all $i \in \{2, 3, \dots, k-1\}$ as $d(a_{i+1})$ is a parameter that we need to know for finding a representation.

Throughout the case analysis we will assume without loss of generality that a_{i-1} is an upper vertex. Also assume that $a_{i'}$ be the maximum $i' < i$ such that $d(a_{i'}) \geq 3$ ($i-1 = i'$ is possible). Moreover, in any representation the agents intersecting the lower-left corner, the upper-right corner and the lower-right corner of s_{a_i} are renamed as z_1, z_3 and z_4 , respectively. The conditions below are simple conditions for avoiding cycles in the graph.

- (1) If z_1 exists and a_i is a lower vertex, then $|st(z_1)| = 0$ and $lt(z_1)$ is a lower-left shrunk monotone path satisfying conditions of Lemma 3.12.
- (2) If z_1 exists, a_i is an upper vertex and either z_3 or z_4 exists, then $st(z_1)$ is a lower-right shrunk monotone path with $|st(z_1)| \leq 1$ and $lt(z_1)$ is a lower-left shrunk monotone path satisfying conditions of Lemma 3.12.
- (3) If z_1 exists, a_i is an upper vertex and neither z_3 nor z_4 exists, P_{11} is shrunk lower-left monotone satisfying conditions of Lemma 3.12 and P_{12} is shrunk lower-right monotone where $\{P_{11}, P_{12}\} = \{st(z_1), lt(z_1)\}$.
- (4) If z_3 exists with a_i being an upper vertex, then $|st(z_3)| = 0$ and $lt(z_3)$ is an upper-right shrunk monotone path.
- (5) If z_3 exists with a_i being a lower vertex, then z_3 has an upper-left monotone tail P_1 satisfying $|P_1| \leq 1$ and an upper-right monotone tail P_2 with some $\{P_1, P_2\} = \{st(z_3), lt(z_3)\}$.
- (6) If z_4 exists with a_i being an upper vertex and z_1 also exists, then z_4 has a lower-left monotone tail P_1 satisfying $|P_1| \leq 1$ and a lower-right monotone tail P_2 with some $\{P_1, P_2\} = \{st(z_4), lt(z_4)\}$.
- (7) If z_4 exists with a_i being an upper vertex and z_1 does not exist, then z_4 has a lower-left monotone tail P_1 conditions of Lemma 3.12 and a lower-right monotone tail P_2 with some $\{P_1, P_2\} = \{st(z_4), lt(z_4)\}$.
- (8) If z_4 exists with a_i being a lower vertex, then $|st(z_4)| = 0$ and $lt(z_4)$ is a lower-right shrunk monotone path.

In each of the cases listed above, there can be at most $2 \times 2 = 4$ possible ways of in which the agents of a_i can play the role of z_1, z_3, z_4 . And for each agent playing the role of z_j the tails can play the role of P_1, P_2 in 2 different ways for each $j \in \{1, 3, 4\}$. Thus there can be at most $4 \times 2^3 = 32$ possible ways in which a_i and its associates can be represented. Among all possible ways those which satisfies the above conditions, we choose the one for which the leaf of the right-monotone tail of z_3 (z_4 cannot exist) or z_4 (z_3 cannot exist) is minimized with respect to $<_x$. As there are at most a constant number of probes to be made, this is achievable in constant time. Moreover, such a representation, if found, will be called the *optimized representation of a_i and its associates*. Otherwise, T is not a 2SUIG.

3.4.4 Representation of a_k and its associates

The canonical representation of a_k and its associates is relatively simpler. Note that a_k can have at most 3 agents and each of them can have at most 2 tails. Suppose that in any representation the agents intersecting the lower-left corner, the upper-right corner and the lower-right corner of s_{a_k} are renamed as z_1, z_3 and z_4 , respectively. Without loss of generality assume that a_{k-1} is an upper vertex. Then our goal will be to find a representation of a_k and its associates which satisfies the following conditions.

- (1) If z_1 exists with a_k being a lower vertex, then $|st(z_1)| = 0$ and $lt(z_1)$ is a lower-left shrunk monotone path satisfying conditions of Lemma 3.12.
- (2) If z_1 exists with a_k being an upper vertex and z_4 exists, then $|st(z_1)| \leq 1$. Moreover, if $|lt(z_1)| \leq 1$, then $lt(z_1)$ is a lower-right shrunk monotone path and $st(z_1)$ is a lower-left shrunk monotone path satisfying conditions of Lemma 3.12. Otherwise, $st(z_1)$ is a lower-right shrunk monotone path and $lt(z_1)$ is a lower-left shrunk monotone path satisfying conditions of Lemma 3.12.
- (3) If z_1 exists with a_k being an upper vertex, z_4 does not exist and z_3 exists with $|st(z_3)| \geq 1$, then $|st(z_1)| \leq 3$. Moreover, if $|lt(z_1)| \leq 3$, then $lt(z_1)$ is a lower-right shrunk monotone path and $st(z_1)$ is a lower-left shrunk monotone path satisfying conditions of Lemma 3.12. Otherwise, $st(z_1)$ is a lower-right shrunk monotone path and $lt(z_1)$ is a lower-left shrunk monotone path satisfying conditions of Lemma 3.12.
- (4) If z_1 exists with a_k being an upper vertex, z_4 does not exist and z_3 exists with $|st(z_3)| = 0$, then $lt(z_1)$ is a lower-right shrunk monotone path and $st(z_1)$ is a lower-left shrunk monotone path satisfying conditions of Lemma 3.12.
- (5) If z_3 exists with a_k being an upper vertex, then $st(z_3)$ is a lower-right shrunk monotone path and $lt(z_3)$ is an upper-right shrunk monotone path. Moreover, if z_4 exists, then $|st(z_3)| = 0$.
- (6) If a_{k-1} has an upper-right monotone tail P with $|P| \geq 2$ and a_k is a lower vertex, then z_3 cannot exist.
- (7) If z_3 exists with a_k being a lower vertex and z_4 exists with $|lt(z_4)| \geq 1$, then z_3 has an upper-left monotone tail P_1 satisfying $|P_1| \leq 1$ and an upper-right monotone tail P_2 with some $\{P_1, P_2\} = \{st(z_3), lt(z_3)\}$.
- (8) If z_3 exists with a_k being a lower vertex and if either z_4 exists with $|lt(z_4)| = 0$ or z_4 does not exist, then $lt(z_3)$ is an upper-right monotone tail and $st(z_3)$ is a lower-right monotone tail.
- (9) If z_4 exists with a_k being an upper vertex and z_3 exists with $|lt(z_3)| \geq 1$, then z_4 has a lower-left monotone tail P_1 satisfying conditions of Lemma 3.12 and a lower-right monotone tail P_2 with some $\{P_1, P_2\} = \{st(z_4), lt(z_4)\}$.
- (4) If z_4 exists with a_k being an upper vertex and if either z_3 exists with $|lt(z_3)| = 0$ or z_3 does not exist, then $st(z_4)$ is an upper-right monotone tail and $st(z_4)$ is an upper-right monotone tail.
- (5) If z_4 exists with a_k being a lower vertex, then $st(z_4)$ is a lower-right shrunk monotone path. and $lt(z_4)$ is a lower-right shrunk monotone path. Moreover, if either z_3 exists or z_2 does not exist but a_{k-1} has an upper-right monotone tail P with $|P| \geq 4$, then $|st(z_4)| = 0$.

In each of the cases listed above, there can be at most $3! = 6$ possible ways of in which the agents of a_1 can play the role of z_1, z_3, z_4 . For each agent playing the role of z_j the tails can be placed in at most two different ways for each $j \in \{1, 3, 4\}$. Thus there can be at most $6 \times 2^3 = 48$ possible ways in which a_i and its associates can be represented. If we can find one representation which satisfies the above conditions, we choose that one and call it *optimized representation of a_k and its associates*. Otherwise, T is not a 2SUIG. As there are at most a constant number of probes to be made, this is achievable in constant time.

3.5 Algorithm

Finally we will describe the algorithm for recognizing if a given tree T is a 2SUIG. Whenever our algorithm concludes that the given tree T is not a 2SUIG, there is a configuration responsible for it. These configurations are forbidden configurations for 2SUIG trees.

- (1) Check if maximum degree of T is at most 4. If not, then T is not a 2SUIG by Lemma 3.1. Otherwise, go to the next step.
- (2) Check if there at most one branch vertex in T . If yes, then T is a 2SUIG by Lemma 3.4. Otherwise, go to the next step.
- (3) Find out the graph induced by the red edges of the tree. If that graph has at least one edge but not a path, then T is not a 2SUIG by Lemma 3.2. Otherwise, go to the next step.
- (4) Find out a (but for some trivial cases it is unique) extended red path $A = a_1 a_2 \dots a_k$. Assign $x_{a_i} = i$ for all $i \in \{1, 2, \dots, k\}$. Moreover, put s_{a_1} in the lower stab.
- (5) For $i = 1$ to k find out the optimized representation of a_i and its associates. If we fail to find such a representation for some $i \in \{1, 2, \dots, k\}$, then T is not a 2SUIG.

Correctness of the algorithm implies from the previous results and discussions. Given a tree it is possible to find out its set of red edges in linear time using post-order traversal. For the other steps we need to probe at most a constant number cases for each red vertex. Thus, it is possible to run the algorithm in $O(|V|)$ time.

4 Conclusion

In this paper we consider the problem of recognizing 2SUIG trees. While doing that we proved a number of structural properties and provided insights regarding how a canonical 2SUIG representation of a tree can be obtained. Recall our discussion on red edges and red vertices of a tree. Observe that, if the red vertices induce a path, then the tree has a unit square intersection representation. Hence, we hope our work can be extended for “k-stab unit interval graphs” and will help solving the tree recognition problem for cubicity two graphs. Moreover, our graph class is obtained by putting edges between two unit

interval graphs. From applications point of view, as unit interval graphs have wide range of applications, our graph class might be able to capture interactions between two unit interval graphs and prove to be valuable in future.

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